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FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING--XII: COMPLEX OPERATORS AND MIN-MAX VARIATION

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SUMMARY

In previous papers, we have applied the functional equation approach of dynamic programming to the study of variational problems associated with the Sturm—Liouville equation of second order with real coefficients. In this way, we were able to obtain the dependence of the Green's function upon the interval length. From this we obtained the corresponding dependence of the characteristic values and the characteristic functions, and similar results for vector-matrix systems.

To apply the same general techniques to the study of equations with complex coefficients, ~~we use~~ min-max variation. It is shown that this method can be applied rigorously.

FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING--XII: COMPLEX OPERATORS AND MIN-MAX VARIATION

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1. Introduction

In [1], we applied the functional equation approach of dynamic programming [2] to the study of the variational problem associated with the Sturm-Liouville equation

$$(1) \qquad (p(x)u')' + (r(x) + \lambda g(x))u = 0, \quad u(a) = u(1) = 0,$$

where p, q, and r are real. In this way, we were able to obtain the dependence of the Green's function upon the end-point a. From this result, we obtained the dependence of the characteristic values and characteristic functions upon a. Corresponding results, the Hadamard variational formulas, were presented for partial differential equations in [3].

In order to apply the functional equation technique, we were required to assume that p, q, and r were real. In [4], we indicated briefly how min-max techniques could be used to study the corresponding problems for complex functions. In this paper, we wish to present the full argument for a particular class of equations of the foregoing form.

2. Boundary Value Problem

Let us consider the following boundary value problem:

$$(1) \qquad U''(x) + q(x)U(x) = v(x), \quad (a < x < b),$$

$$U(a) = 0, \quad U(b) = 0,$$

where $q(x)$ and $v(x)$ are complex-valued functions continuous on the closed interval $[a,b]$. If the Sturm-Liouville problem obtained by replacing q by λq does not have $\lambda = 1$ as a characteristic value, then the boundary value problem has a unique solution given by the equation

$$(2) \qquad U(x) = \int_a^b K(x,y,a)v(y)dy$$

where $K(x,y,a)$ is the Green's function of the problem. We shall regard b as fixed and study the dependence of the Green's function upon a.

In order to use an approach suggested by dynamic programming, we shall find it convenient to study the system

$$(3) \qquad u''(x) + q(x)u(x) = v(x), \quad (a < x < b),$$

$$u(a) = c, \quad u(b) = 0.$$

A solution of this system can, under the above assumptions, be expressed in the form

$$(4) \qquad u(x) = U(x) + c\phi(x)$$

where

$$(5) \qquad \phi''(x) + q(x)\phi(x) = 0, \quad (a < x < b),$$

$$\phi(a) = 1, \quad \phi(b) = 0.$$

Since the problem (3) is associated with a complex linear differential operator which is symmetric, we can obtain a

corresponding variational problem. Let $u(x) = u_1(x) + iu_2(x)$, $q(x) = q_1(x) + iq_2(x)$, $v(x) = v_1(x) + iv_2(x)$, with $u_1, u_2, q_1, q_2, v_1, v_2$ real-valued functions, and let $c = c_1 + ic_2$ with c_1 and c_2 real numbers. The system (3) is then equivalent to the real system

$$\begin{aligned} (6) \qquad & u_1''(x) + q_1(x)u_1(x) - q_2(x)u_2(x) = v_1(x), \\ & u_2''(x) + q_1(x)u_2(x) + q_2(x)u_1(x) = v_2(x), \\ & u_1(a) = c_1, \quad u_2(a) = c_2, \quad u_1(b) = u_2(b) = 0. \end{aligned}$$

We consider the integral

$$J(u_1,u_2) = \int_a^b \Big\{ - u_1'^2 + u_2'^2 + q_1u_1^2 - 2q_2u_1u_2 - q_1u_2^2 - 2u_1v_1 + 2u_2v_2 \Big\} dx$$

and consider the variational problem of determining

$$(7) \qquad f(a,c_1,c_2) = \max_{u_1} \min_{u_2} J(u_1,u_2),$$

where the maximum is taken over piecewise continuously differentiable functions u_1 for which $u_1(a) = c_1$, $u_1(b) = 0$, and the minimum is taken over piecewise continuously differentiable functions u_2 for which $u_2(a) = c_2$, $u_2(b) = 0$.

The equations (6) are the Euler equations for the variational problem (7) and the functions u_1 and u_2 yielding the max min, if they exist, provide a solution of the system (6). To see this, we let $\eta_1(x)$ and $\eta_2(x)$ be arbitrary piecewise

continuously differentiable functions for which $\eta_1(a) = \eta_2(a) = \eta_1(b) = \eta_2(b) = 0$ and let ϵ_1 and ϵ_2 be positive real numbers. Then

$$J(u_1 + \epsilon_1\eta_1, u_2 + \epsilon_2\eta_2) = J(u_1,u_2) + 2\epsilon_1 A_1 + 2\epsilon_2 A_2 + \epsilon_1^2 A_{11} + 2\epsilon_1\epsilon_2 A_{12} + \epsilon_2^2 A_{22},$$

where

$$A_1 = \int_a^b (-u_1'\eta_1' + q_1u_1\eta_1 - q_2u_2\eta_1 - v_1\eta_1)dx,$$

$$A_2 = \int_a^b (u_2'\eta_2' - q_2u_1\eta_2 - q_1u_2\eta_2 + v_2\eta_2)dx,$$

$$A_{11} = \int_a^b (-\eta_1'^2 + q_1\eta_1^2)dx,$$

$$A_{12} = -\int_a^b q_2\eta_1\eta_2 dx,$$

$$A_{22} = \int_a^b (\eta_2'^2 - q_1\eta_2^2)dx.$$

Integrating by parts we obtain

$$A_1 = \int_a^b \eta_1(x)\left[u_1''(x) + q_1(x)u_1(x) - q_2(x)u_2(x) - v_1(x)\right]dx,$$

$$A_2 = -\int_a^b \eta_2(x)\left[u_2''(x) + q_1(x)u_2(x) + q_2(x)u_1(x) - v_2(x)\right]dx.$$

Consequently (u_1,u_2) is a stationary point for J if and only if u_1 and u_2 yield a solution of the boundary value problem (6).

In order to assure the existence of the max min appearing in (7), it is sufficient to assume that $q_1(x) < 0$ for $a < x < b$. It then follows that if $\eta_1(x) \neq 0$ and $\eta_2(x) \neq 0$, then $A_{11} < 0$ and $A_{22} > 0$. Consequently, the max min in (7) will exist and also

(8) $$f(a, c_1, c_2) = \min_{u_2} \max_{u_1} J(u_1, u_2)$$

where the minimum and maximum are taken over the same classes of functions as for (7).

We remark that if $q_1(x)$ is not negative on the entire interval $[a, b]$ then one can employ analytic continuation as in [4] to reduce the problem to one for which the variational argument is applicable.

3. Interchanging min and max Operators

Let X_1, X_2, Y_1, and Y_2 be sets; and let x_1, x_2, y_1, and y_2 be variables ranging over these sets. Let $\Phi(x_1, x_2, y_1, y_2)$ be a real-valued function defined for all values of the variables. Assuming that all of the maxima and minima mentioned exist, we shall prove the following lemma.

LEMMA 1. If

(1) $$\rho = \max_{x_1} \max_{x_2} \min_{y_1} \min_{y_2} \Phi(x_1, x_2, y_1, y_2)$$

$$= \min_{y_1} \min_{y_2} \max_{x_1} \max_{x_2} \Phi(x_1, x_2, y_1, y_2),$$

then

$$\max_{x_1} \min_{y_1} \max_{x_2} \min_{y_2} \Phi(x_1, x_2, y_1, y_2) = \rho \tag{2}$$

and

$$\min_{y_1} \max_{x_1} \min_{y_2} \max_{x_2} \Phi(x_1, x_2, y_1, y_2) = \rho. \tag{3}$$

Proof. We use repeatedly the fact that

$$\max_{x} \min_{y} \Psi(x,y) \leq \min_{y} \max_{x} \Psi(x,y) \tag{4}$$

for any function Ψ of variables x and y ranging over sets X and Y, provided the maxima and minima exist. (See [5, p.10].) We have

$$\begin{aligned} \max_{x_1} \max_{x_2} \min_{y_1} \min_{y_2} \Phi &\leq \max_{x_1} \min_{y_1} \max_{x_2} \min_{y_2} \Phi \\ &\leq \min_{y_1} \max_{x_1} \max_{x_2} \min_{y_2} \Phi \\ &\leq \min_{y_1} \max_{x_1} \min_{y_2} \max_{x_2} \Phi \\ &\leq \min_{y_1} \min_{y_2} \max_{x_1} \max_{x_2} \Phi. \end{aligned} \tag{5}$$

By assumption the two expressions at the ends of this string of inequalities are equal. Hence we can replace "$\leq$" throughout by "=" and the lemma follows.

We remark that the proof shows that the lemma holds even if the set X_2 depends upon the variable x_1 and the set Y_2 depends upon the variable y_1.

Lemma 1 can be regarded as a statement of "the principle of optimality" for a particular type of two-stage game. In this interpretation ρ is the value of the two person game. The first player chooses the strategy x_1 for the first stage and the strategy x_2 for the second stage. The other player chooses y_1 for the first stage and y_2 for the second. For a further discussion of the principle of optimality for multi-stage games we refer to [2;p.29].

4. Application of Dynamic Programming

Using an approach suggested by the theory of dynamic programming, we shall discuss a variational problem of which the problem of §2 is a special case. Let F be a continuous function of five variables. Let

$$(1) \qquad f(a,c_1,c_2) = \max_{x_1} \min_{y_1} \int_a^b F(x(t),x_1(t),y(t),y_1(t),t)dt$$

$$= \min_{y_1} \max_{x_1} \int_a^b F(x(t),x_1(t),y(t),y_1(t),t)dt$$

where

$$(2) \qquad x(t) = c_1 + \int_a^t x_1(s)ds,$$

$$y(t) = c_2 + \int_a^t y_1(s)ds,$$

and the maxima are taken over functions $x_1(t)$ which are piecewise continuous on the interval $[a,b]$ and the minima are taken over functions $y_1(t)$ which are piecewise continuous on

the interval $[a,b]$. The assumption that the max min is equal to the min max will be important in the following discussion. In addition we shall assume that the function f is continuously differentiable and that the functions x_1 and y_1 yielding the min max are continuous.

Let Δ be a small positive number. We let $\max_I$ denote the maximum taken over piecewise continuous functions x_1 restricted to the interval $[a, a + \Delta]$, $\max_{II}$ denote the maximum over piecewise continuous functions x_1 restricted to the interval $[a + \Delta, b]$, and similarly $\min_I$ denote the minimum over piecewise continuous y_1 on $[a, a + \Delta]$, and $\min_{II}$ the minimum over piecewise continuous y_1 on $[a + \Delta, b]$. We have by Lemma 1

$$(3) \qquad f(a,c_1,c_2) = \max_{I} \max_{II} \min_{I} \min_{II} \int_a^b F dt$$

$$= \max_{I} \min_{I} \max_{II} \min_{II} \left\{ \int_a^{a+\Delta} F dt + \int_{a+\Delta}^b F dt \right\}$$

$$= \max_{I} \min_{I} \left\{ \int_a^{a+\Delta} F dt + \max_{II} \min_{II} \int_{a+\Delta}^b F dt \right\}$$

$$= \max_{I} \min_{I} \left\{ \int_a^{a+\Delta} F dt + f(a + \Delta, x(a + \Delta), y(a + \Delta)) \right\}.$$

This states algebraically the principle of optimality for multistage games--at each stage an optimal policy is one leading to an optimal continuation.

Using the assumed continuity of F and continuous differentiability of f, we obtain for $\Delta \rightarrow 0$

$$(4)\qquad f(a,c_1,c_2) = \max_{I}\ \min_{I} \Big\{\Delta F(c_1,x_1(a),c_2,x_2(a),a) + f(a,c_1,c_2) + \Delta \frac{\partial f}{\partial a}(a,c_1,c_2) + \Delta x_1(a)\frac{\partial f}{\partial c_1}(a,c_1,c_2) + \Delta y_1(a)\frac{\partial f}{\partial c_2}(a,c_1,c_2) + o(\Delta)\Big\},$$

and hence for $\Delta \rightarrow 0$

$$(5)\qquad 0 = \max_{I}\ \min_{I} \Big\{F(c_1,x_1(a),c_2,x_2(a),a) + \frac{\partial f}{\partial a} + x_1(a)\frac{\partial f}{\partial c_1} + y_1(a)\frac{\partial f}{\partial c_2} + o(1)\Big\}.$$

Here it is tempting to ignore the $o(1)$ term and say that in the limit a choice of x_1 and y_1 over the interval $[a,a + \Delta]$ amounts to a choice of $x_1(a)$ and $y_1(a)$. Let us justify this argument. Let $\beta(t)$ be an arbitrary continuous function on the interval $[a,b]$ such that $\beta(a) = 0$. Since we are assuming the existence of a continuous solution, let us take the maximum and minimum not over all piecewise continuous functions, but over all continuous functions of the form

$$(6)\qquad \begin{aligned} x_1(t) &= x_1(a) + \hat{x}_1(t), \quad |x_1(t)| \leq \beta(t), \\ y_1(t) &= y_1(a) + \hat{y}_1(t), \quad |y_1(t)| \leq \beta(t). \end{aligned}$$

Clearly, by choosing β appropriately we can have the solution of our original problem in the admissible class.

Abbreviating by setting $c_1{}' = x_1(a)$, $c_2{}' = y_1(a)$ and interchanging minimum and maximum operators according to Lemma 1, we have for $\Delta \rightarrow 0$

$$(7)\qquad 0 = \max_{c_1'}\ \max_{\hat{x}_1(t)}\ \min_{c_2'}\ \min_{\hat{y}_1(t)} \Big\{F(c_1,c_1',c_2,c_2',a) + \frac{\partial f}{\partial a} + c_1'\frac{\partial f}{\partial c_1} + c_2'\frac{\partial f}{\partial c_2} + o(1)\Big\}$$

$$= \max_{c_1'}\ \min_{c_2'} \Big\{F(c,c_1',c_2,c_2',a) + \frac{\partial f}{\partial a} + c_1'\frac{\partial f}{\partial c_1} + c_2'\frac{\partial f}{\partial c_2} + \max_{\hat{x}_1(t)}\ \min_{\hat{y}_1(t)}\ o(1)\Big\}.$$

Because of the assumed properties of the solution, we can restrict the point (c_1',c_2') to a bounded region. Then, if we investigate the $o(1)$ term more closely, we see that the restrictions (6) guarantee that the term $o(1)$ is <u>uniformly</u> small for all functions x_1 and y_1 for which $|\hat{x}_1(t)| \le \beta(t)$, $|\hat{x}_2(t)| \le \beta(t)$. Consequently, we obtain

$$(8)\qquad 0 = \max_{c_1'}\ \min_{c_2'} \Big\{F(c_1,c_1',c_2,c_2',a) + \frac{\partial f}{\partial a}(a,c_1,c_2) + c_1'\frac{\partial f}{\partial c_1}(a,c_1,c_2) + c_2'\frac{\partial f}{\partial c_2}(a,c_1,c_2)\Big\}.$$

Specializing this result to the variational problem of §2, we have

$$(9)\qquad 0 = \max_{c_1'}\ \min_{c_2'} \Big\{- c_1'^2 + c_2'^2 + c_1^2 q_1(a) - 2c_1c_2q_2(a) - c_2^2 q_1(a) - 2c_1v_1(a) + 2c_2v_2(a) + \frac{\partial f}{\partial a}(a,c_1,c_2) + c_1'\frac{\partial f}{\partial c_1}(a,c_1,c_2) + c_2'\frac{\partial f}{\partial c_2}(a,c_1,c_2)\Big\}.$$

It is easily seen that the max min of this quadratic function is attained for

$$(10) \qquad c_1' = \frac{1}{2}\frac{\partial f}{\partial c_1}, \quad c_2' = -\frac{1}{2}\frac{\partial f}{\partial c_2}.$$

Substituting these values into (9) we obtain the partial differential equation

$$(11) \qquad -\frac{\partial f}{\partial a}(a,c_1,c_2) = \frac{1}{4}\left(\frac{\partial f}{\partial c_1}(a,c_1,c_2)\right)^2 - \frac{1}{4}\left(\frac{\partial f}{\partial c_2}(a,c_1,c_2)\right)^2$$
$$- 2c_1v_1(a) + 2c_2v_2(a)$$
$$+ (c_1{}^2 - c_2{}^2)q_1(a) - 2c_1c_2q_2(a).$$

5. Alternate Approach

Another way of formulating the variational problem of §2 is gotten by observing that

$$(1) \qquad J(u_1,u_2) = \mathrm{Re}\int_a^b \Big\{-(u'(x))^2 + q(x)(u(x))^2$$
$$- 2u(x)v(x)\Big\}\,dx.$$

Similarly the partial differential equation (4.11) can be written

$$(2) \qquad -\frac{\partial f}{\partial a}(a,c_1,c_2) = \frac{1}{4}\left(\frac{\partial f}{\partial c_1}\right)^2 - \frac{1}{4}\left(\frac{\partial f}{\partial c_2}\right)^2 + \mathrm{Re}\{-2cv(a) + c^2q(a)\},$$

where $c = c_1 + ic_2$.

The following lemma will be used in the next section.

LEMMA 2. *If* $F(x,y)$ *is continuous for* $a < x < b$, $a < y < b$, *if* $F(x,y) = F(y,x)$, *and if*

$$(3) \qquad \mathrm{Re}\int_a^b\int_a^b F(x,y)v(x)v(y)\,dxdy = 0$$

<u>for every continuous function</u> v, <u>then</u> $F(x,y)$ <u>vanishes identically</u>.

<u>If</u> $F(x)$ <u>is continuous for</u> $a < x < b$, <u>and if</u>

$$\mathrm{Re}\int_a^b F(x)v(x)dx = 0$$

<u>for every continuous function</u> v, <u>then</u> $F(x)$ <u>vanishes identically</u>.

<u>Proof</u>. In case F is a real-valued function and (3) holds for every real-valued continuous v, the first assertion of the lemma was proved in [1]. Let $F(x,y) = F_1(x,y) + iF_2(x,y)$, $v(x) = v_1(x) + iv_2(x)$ with F_1, F_2, v_1, v_2 real-valued functions. We have

$$(4) \qquad 0 = \mathrm{Re}\int_a^b\int_a^b F(x,y)v(x)v(y)dxdy$$

$$= \int_a^b\int_a^b \Big\{F_1(x,y)\big[v_1(x)v_1(y) - v_2(x)v_2(y)\big]$$

$$- F_2(x,y)\big[v_1(x)v_2(y) + v_2(x)v_1(y)\big]\Big\}dxdy.$$

Taking $v_2(x) \equiv 0$, we obtain

$$\int_a^b\int_a^b F_1(x,y)v_1(x)v_1(y)dxdy = 0$$

for all continuous real-valued functions v_1. Hence by the lemma of [1], $F_1(x,y) \equiv 0$. Next, taking $v_2(x) \equiv v_1(x)$, we obtain

$$(5) \qquad -2\int_a^b\int_a^b F_2(x,y)v_1(x)v_1(y)dxdy = 0,$$

and thus $F_2(x,y) \equiv 0$. This completes the proof of the first assertion of the lemma. The second assertion can be obtained as a corollary by taking $F(x,y) = iF(x)F(y)$.

6. Variation of the Green's Function

Now let us use the partial differential equation (5.2) to study the variation of the Green's function $K(x,y,a)$ with a. Let $u(x) = u_1(x) + iu_2(x)$ be the function which yields the max min in (2.7) for given a and c. Writing $u = U + c\phi$ and integrating by parts, we find

(1) $$-\int_a^b u'^2 dx = \int_a^b \left\{U(U'' + 2c\phi'') + c^2\phi\phi''\right\}dx + c^2\phi'(a).$$

Hence, using the differential equations $U'' + qU = v$ and $\phi'' + q\phi = 0$, we obtain

(2) $$f(a,c_1,c_2) = \mathrm{Re}\int_a^b (-u'^2 + qu^2 - 2uv)dx$$

$$= \mathrm{Re}\left\{-\int_a^b U(x)v(x)dx - 2c\int_a^b \phi(x)v(x)dx + c^2\phi'(a)\right\},$$

and hence by (2.2)

(3) $$f(a,c_1,c_2) = \mathrm{Re}\left\{-\int_a^b\int_a^b K(x,y,a)v(x)v(y)dxdy - 2c\int_a^b \phi(x)v(x)dx + c^2\phi'(a)\right\}.$$

Using the differential equation (5.2), we obtain upon equating terms independent of c_1 and c_2,

$$\mathrm{Re}\int_a^b\int_a^b \frac{\partial K}{\partial a}(x,y,a)v(x)v(y) = \left(\mathrm{Re}\int_a^b \phi(x)v(x)dx\right)^2$$

$$- \left(\mathrm{Im}\int_a^b \phi(x)v(x)dx\right)^2$$

$$= \mathrm{Re}\int_a^b\int_a^b \phi(x)\phi(y)v(x)v(y)dxdy$$

for every continuous function v. Because of the symmetry of the Green's function, we can apply Lemma 2 to obtain

$$\frac{\partial K}{\partial a}(x,y,a) = \phi(x)\phi(y).$$

Also

$$\int_a^b \phi v dx = \int_a^b \phi(U'' + qU)dx = \int_a^b U(\phi'' + q\phi)dx - U'(a)\phi(a)$$

$$= - U'(a),$$

and hence

$$(4) \qquad \int_a^b \phi(y)v(y)dy = -\int_a^b \frac{\partial K}{\partial x}(a,y,a)v(y)dy,$$

for every continuous function v. Hence, by Lemma 2,

$$(5) \qquad \phi(y) = - \frac{\partial K}{\partial x}(a,y,a).$$

Thus

$$(6) \qquad \frac{\partial K}{\partial a}(x,y,a) = \frac{\partial K}{\partial x}(a,y,a)\frac{\partial K}{\partial y}(x,a,a).$$

7. Further Results

Having obtained the bacic variational formula of (6.6), we can apply it to the equation

$$(1) \qquad u''(x) + (p(x) + \lambda q(x))u(x) = v(x), \quad u(a) = c, \quad u(1) = 0,$$

to obtain an equation for the resolvent as a function of a. Specializing the values of λ as in [1], we obtain in this way variational formulas for the characteristic values and characteristic functions.

Finally, let us mention that as in [1] it is easy to obtain analogous results for the vector-matrix version of (1).

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